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Financial Modelling Using Discrete Stochastic Calculus

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ABSTRACT: In the present report, a review of discrete calculus on directed graphs is presented. It is found that the binary tree is a special directed graph that contains both the exterior calculus and stochastic calculus as different continuum limits are taken. In the latter case, we arrive at something that may be referred to as "discrete stochastic calculus." The resulting discrete stochastic calculus may be applied to any stochastic financial model and is guaranteed to produce results that converge in the continuum limit. Discrete stochastic calculus is applied to the Black-Scholes model for an illustration. The resulting algorithm generated by discrete stochastic calculus agrees with that of the Cox-Ross-Rubinstein model, as it should. The results presented here are preliminary and are intended to encourage others to learn discrete stochastic calculus and apply it to more complicated financial models.

Contents

1. A Brief Mathematical Review

In the present paper, we will be concerned with quantities that can be integrated over p-dimensional domains. Although the material we discuss generalizes for any p, we will restrict attention to the cases of $p = 0$ and $p = 1$.

1.1 0-Forms

What does it mean to integrate something over a 0-dimensional domain, i.e. a point?

Consider some arbitrary space $\mathcal{M}$ and a function $f : \mathcal{M} \to \mathbb{R}$, i.e. to each point $p \in \mathcal{M}$ we associate a real number $f(p) \in \mathbb{R}$. We can formally define a 0-dimensional integral over the point p via

$$\int_p f = f(p), \tag{1.1}$$

i.e. integration of a function f over a 0-dimensional domain p is simply evaluation of the function at the point. The space of all such integrands constitute a vector space since

$$\int_p (af + bg) = a \int_p f + b \int_p g \tag{1.2}$$

for all $a, b \in \mathbb{R}$. When viewed in this way as the space of 0-dimensional integrands, the space of functions is referred to as the space of 0-forms, i.e. a 0-form is simply a scalar function viewed as the integrand of a 0-dimensional integral. The space of 0-forms on $\mathcal{M}$ will be denoted $\Omega^0(\mathcal{M})$, or simply Ω^0 if $\mathcal{M}$ is understood.

1.2 1-Forms

If the space $\mathcal{M}$ above admits domains that may be thought of 1-dimensional oriented segments or curves, then we can formally define the space $C_1(\mathcal{M})$ of all such curves in $\mathcal{M}$. Now consider a function $\alpha : C_1(\mathcal{M}) \to \mathbb{R}$, i.e. to each curve $\gamma \in C_1(\mathcal{M})$ we associate a real number $\alpha(\gamma) \in \mathbb{R}$. As before, we can write down a formal 1-dimensional integral via

$$\int_\gamma \alpha = \alpha(\gamma). \tag{1.3}$$

The space of all such functions is referred to as the space of 1-forms and is denoted $\Omega^1(\mathcal{M})$.

Admittedly, this may seem like an odd way to view something (integration) that is already quite familiar, but the point here is to highlight the duality between p-dimensional domains and p-dimensional functionals on that domain.

1.3 Exterior Derivative

The exterior derivative is a map $d : \Omega^0 \to \Omega^1$ satisfying

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g) \tag{1.4}$$

and

$$\int_\gamma \mathrm{d}f = \int_{\partial\gamma} f, \tag{1.5}$$

where $\partial : C_1 \to C_0$ is the boundary map. For example, if $\gamma \in C_1$ is an oriented curve beginning at $p \in C_0$ and ending at $q \in C_0$, then

$$\partial\gamma = q - p. \tag{1.6}$$

It follows that

$$\begin{aligned}\int_\gamma \mathrm{d}f &= \int_{\partial\gamma} f \\ &= \int_{q-p} f \\ &= f(q) - f(p).\end{aligned} \tag{1.7}$$

This is simply a statement of the fundamental theorem of calculus in a slightly more geometric language.

2. Discrete Calculus

In order to develop robust numerical algorithms, we would like to have discrete versions of 0-forms, 1-forms, integration, and the exterior derivative available to us. Fortunately, most of these were provided by Poincare nearly 100 years ago.

2.1 Discrete 0-Forms

In a discrete space, the points are countable, i.e. can be indexed by integers. This is in contrast to a continuum space where the points are infinite but not countable. We will label these discrete points by $\mathbf{e}_\kappa$. We can define discrete integration over these points in precisely the same way as in the continuum, i.e.

$$\int_{\mathbf{e}_\kappa} f = f(\kappa), \tag{2.1}$$

where $f(\kappa) \in \mathbb{R}$ is the value of a discrete 0-form $f \in \Omega^0$ at the node $\mathbf{e}_\kappa$. Due to the countable nature of the discrete space, this allows us to express the discrete 0-form f as a summation

$$f = \sum_\lambda f(\lambda)\mathbf{e}^\lambda, \tag{2.2}$$

where $\mathbf{e}^\lambda$ is a basis for discrete 0-forms defined by

$$\int_{\mathbf{e}_\kappa} \mathbf{e}^\lambda = \delta_{\kappa,\lambda}. \tag{2.3}$$

It may be helpful to see this worked out explicitly. Hence, plugging (2.2) into an integral over $\mathbf{e}_\kappa$ results in

$$\begin{aligned}\int_{\mathbf{e}_\kappa} f &= \sum_\lambda f(\lambda) \int_{\mathbf{e}_\kappa} \mathbf{e}^\lambda \\ &= \sum_\lambda f(\lambda)\delta_{\kappa,\lambda} \\ &= f(\kappa)\end{aligned} \tag{2.4}$$

as required.

In addition to simply expressing discrete 0-forms in terms of a basis, this also allows us to develop the algebra of discrete 0-forms via simple algebraic rules. To multiply two discrete 0-forms, it suffices to define multiplication of the basis elements. This is given simply by

$$\mathbf{e}^\kappa \mathbf{e}^\lambda = \delta_{\kappa,\lambda} \mathbf{e}^\kappa. \tag{2.5}$$

With this simple rule, we can explicitly compute

$$\begin{aligned} fg &= \left(\sum_\kappa f(\kappa)\mathbf{e}^\kappa\right)\left(\sum_\lambda g(\lambda)\mathbf{e}^\lambda\right) \\ &= \sum_{\kappa,\lambda} f(\kappa)g(\lambda)\delta_{\kappa,\lambda}\mathbf{e}^\kappa \\ &= \sum_\kappa f(\kappa)g(\kappa)\mathbf{e}^\kappa, \end{aligned} \tag{2.6}$$

which is also as expected.

2.2 Discrete 1-Forms

In addition to discrete points $\mathbf{e}_\kappa$, we will have discrete directed edges $\mathbf{e}_{\kappa\lambda}$ extending from the point $\mathbf{e}_\kappa$ to the point $\mathbf{e}_\lambda$. As before, we can define discrete integration via

$$\int_{\mathbf{e}_{\kappa\lambda}} \alpha = \alpha_\lambda(\kappa), \tag{2.7}$$

which motivates the introduction of basis 1-forms $\mathbf{e}^{\mu\nu}$ such that

$$\alpha = \sum_{\mu\nu} \alpha_\nu(\mu)\mathbf{e}^{\mu\nu}, \tag{2.8}$$

where

$$\int_{\mathbf{e}_{\kappa\lambda}} \mathbf{e}^{\mu\nu} = \delta_{\kappa,\mu}\delta_{\lambda,\nu}. \tag{2.9}$$

Now, what is critically distinct about discrete 0- and 1-forms compared to the continuum is that discrete 0-forms and 1-forms do not commute. To see this, again it suffices to consider the basis elements. The multiplication rules for discrete 0-forms and discrete 1-forms are given by

$$\mathbf{e}^{\kappa\lambda}\mathbf{e}^\mu = \delta_{\lambda,\mu}\mathbf{e}^{\kappa\lambda} \tag{2.10}$$

and

$$\mathbf{e}^\mu \mathbf{e}^{\kappa\lambda} = \delta_{\mu,\kappa}\mathbf{e}^{\kappa\lambda}. \tag{2.11}$$

To help illustrate the simple geometric meaning of this multiplication rule, it helps to consider an arbitrary discrete 0-form and a single discrete 1-form basis element. In this case we have,

$$
\begin{aligned}
f\mathbf{e}^{\kappa\lambda} &= \left(\sum_\mu f(\mu)\mathbf{e}^\mu\right)\mathbf{e}^{\kappa\lambda} \\
&= \sum_\mu f(\mu)\delta_{\mu,\kappa}\mathbf{e}^{\kappa\lambda} \\
&= f(\kappa)\mathbf{e}^{\kappa\lambda}. \qquad (2.12)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\mathbf{e}^{\kappa\lambda} f &= \mathbf{e}^{\kappa\lambda}\left(\sum_\mu f(\mu)\mathbf{e}^\mu\right) \\
&= \sum_\mu f(\mu)\delta_{\lambda,\mu}\mathbf{e}^{\kappa\lambda} \\
&= f(\lambda)\mathbf{e}^{\kappa\lambda}. \qquad (2.13)
\end{aligned}
$$

That is, if you multiply a discrete 0-form on the *left*, it picks out the value of the discrete 0-form at the *beginning* of the directed edge. If you multiply on the *right*, it picks out the value of the discrete 0-form at the *end* of the directed edge. If you are familiar with stochastic calculus, this might make you think of the differences in the definitions of the Stratonovich and Ito integrals. It is no coincidence!

2.3 Discrete Exterior Derivative

The discrete exterior derivative is easy to write down and is given by

$$
d\mathbf{e}^\kappa = \sum_\lambda \left(\mathbf{e}^{\lambda\kappa} - \mathbf{e}^{\kappa\lambda}\right). \qquad (2.14)
$$

This may be derived via the fundamental theorem of discrete calculus

$$
\begin{aligned}
\int_{\mathbf{e}_{\lambda\mu}} d\mathbf{e}^\kappa &= \int_{\partial\mathbf{e}_{\lambda\mu}} \mathbf{e}^\kappa \\
&= \int_{\mathbf{e}_\mu - \mathbf{e}_\lambda} \mathbf{e}^\kappa \\
&= \delta_{\kappa,\mu} - \delta_{\kappa,\lambda}. \qquad (2.15)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d}\mathbf{e}^{\kappa} &= \sum_{\lambda\mu} \left(\int_{\mathbf{e}_{\lambda\mu}} \mathrm{d}\mathbf{e}^{\kappa} \right) \mathbf{e}^{\lambda\mu} \\
&= \sum_{\lambda\mu} \left(\delta_{\kappa,\mu} - \delta_{\kappa,\lambda}\right) \mathbf{e}^{\lambda\mu} \\
&= \sum_{\lambda} \mathbf{e}^{\lambda\kappa} - \sum_{\mu} \mathbf{e}^{\kappa\mu} \\
&= \sum_{\lambda} \left(\mathbf{e}^{\lambda\kappa} - \mathbf{e}^{\kappa\lambda} \right).
\end{aligned} \tag{2.16}
$$

If we were to think of the $\mathbf{e}^{\kappa\lambda}$ as directed edges (technically, they are dual to directed edges), then this says that the discrete exterior derivative of the node $\mathbf{e}^{\kappa}$ is the sum of edges directed *into* $\mathbf{e}^{\kappa}$ minus the sum of the edges directed *away* from $\mathbf{e}^{\kappa}$.

Now we can consider the discrete exterior derivative of an arbitrary discrete 0-form

$$
\begin{aligned}
\mathrm{d}f &= \sum_{\kappa} f(\kappa)\mathrm{d}\mathbf{e}^{\kappa} \\
&= \sum_{\kappa,\lambda} f(\kappa) \left(\mathbf{e}^{\lambda\kappa} - \mathbf{e}^{\kappa\lambda} \right) \\
&= \sum_{\kappa,\lambda} f(\kappa)\mathbf{e}^{\lambda\kappa} - \sum_{\kappa,\lambda} f(\kappa)\mathbf{e}^{\kappa\lambda} \\
&= \sum_{\kappa,\lambda} f(\lambda)\mathbf{e}^{\kappa\lambda} - \sum_{\kappa,\lambda} f(\kappa)\mathbf{e}^{\kappa\lambda} \\
&= \sum_{\kappa,\lambda} \left[f(\lambda) - f(\kappa)\right] \mathbf{e}^{\kappa\lambda},
\end{aligned} \tag{2.17}
$$

where in the first term of the fourth line, we simply interchanged the summation variables $\kappa \leftrightarrow \lambda$. This same result could have also been obtain by an application of the fundamental theorem of discrete calculus.

To check the product rule, we simply compute both sides. First,

$$
\mathrm{d}(fg) = \sum_{\kappa,\lambda} \left[f(\lambda)g(\lambda) - f(\kappa)g(\kappa)\right] \mathbf{e}^{\kappa\lambda}. \tag{2.18}
$$

Second,

$$
\begin{aligned}
(\mathrm{d}f)g &= \left(\sum_{\kappa,\lambda} \left[f(\lambda) - f(\kappa)\right] \mathbf{e}^{\kappa\lambda} \right) \left(\sum_{\mu} g(\mu)\mathbf{e}^{\mu} \right) \\
&= \sum_{\kappa,\lambda} \left[f(\lambda)g(\lambda) - f(\kappa)g(\lambda)\right] \mathbf{e}^{\kappa\lambda}.
\end{aligned} \tag{2.19}
$$

Third,

$$f(\mathrm{d}g) = \left(\sum_{\mu} f(\mu)\mathbf{e}^{\mu}\right)\left(\sum_{\kappa,\lambda}\left[g(\lambda) - g(\kappa)\right]\mathbf{e}^{\kappa\lambda}\right)$$
$$= \sum_{\kappa,\lambda}\left[f(\kappa)g(\lambda) - f(\kappa)g(\kappa)\right]\mathbf{e}^{\kappa\lambda}. \tag{2.20}$$

Summing the previous two results, we obtain

$$(\mathrm{d}f)g + f(\mathrm{d}g) = \sum_{\kappa,\lambda}\left[f(\lambda)g(\lambda) - f(\kappa)g(\kappa)\right]\mathbf{e}^{\kappa\lambda}, \tag{2.21}$$

thus verifying the product rule

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g). \tag{2.22}$$

Note the crucial role the noncommutativity of discrete 0-forms and discrete 1-forms played in satisfying the product rule. Again, this is no coincidence.

3. The Binary Tree

In the previous Section, we outlined the tools required to develop a discrete calculus. We saw that the basis of the discrete calculus consisted of a set of nodes $\mathbf{e}_\kappa$ and a set of directed edges $\mathbf{e}_{\kappa\lambda}$. However, this is the same information needed to define a directed graph, i.e. we have defined a discrete calculus on an arbitrary directed graph. However, in [?], it was pointed out that certain graphs have more desirable qualities than others. In particular, it was shown that if you desire to model an n-dimensional manifold in the continuum limit, then you must have exactly n edges directed away from each node. A particularly simple case of this is what we called an n-diamond. For the purposes of this paper, we are only concerned with 2-diamonds, which happen to correspond to a binary tree. Therefore, in the remainder of this paper, we will be concerned with directed graphs corresponding to binary trees.

In a binary tree, it makes sense to use multi-indices to describe the location of points. For example, a node located at the ith position in the traverse direction and the jth time step will be denoted by $\mathbf{e}_{(i,j)}$. Consequently, an arbitrary discrete 0-form on a binary tree may be expressed as

$$\phi = \sum_{i,j}\phi(i,j)\mathbf{e}^{(i,j)}. \tag{3.1}$$

Due to the rigid structure of the binary tree, we may express an arbitrary discrete 1-form via

$$\alpha = \sum_{i,j}\alpha_+(i,j)\mathbf{e}^{(i,j)(i+1,j+1)}$$
$$+ \sum_{i,j}\alpha_-(i,j)\mathbf{e}^{(i,j)(i-1,j+1)} \tag{3.2}$$

To make contact with the familiar continuum theory, it is helpful to introduce coordinates

$$t = \sum_{i,j} t(i,j)\mathbf{e}^{(i,j)} \tag{3.3}$$

and

$$x = \sum_{i,j} x(i,j)\mathbf{e}^{(i,j)}, \tag{3.4}$$

where

$$\Delta t = t(i+1,j+1) - t(i,j) = t(i-1,j+1) - t(i,j) \tag{3.5}$$

and

$$\Delta x = x(i+1,j+1) - x(i,j) = -\left[x(i-1,j+1) - x(i,j)\right]. \tag{3.6}$$

With this, we can compute the differentials

$$\begin{aligned} \mathrm{d}t &= \sum_{i,j} \mathrm{d}t(i,j) \\ &= \sum_{i,j} \Delta t \left[\mathbf{e}^{(i,j)(i+1,j+1)} + \mathbf{e}^{(i,j)(i-1,j+1)}\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathrm{d}x &= \sum_{i,j} \mathrm{d}x(i,j) \\ &= \sum_{i,j} \Delta x \left[\mathbf{e}^{(i,j)(i+1,j+1)} - \mathbf{e}^{(i,j)(i-1,j+1)}\right]. \end{aligned} \tag{3.8}$$

Furthermore, with a little algebra we can compute the commutative relations

$$[\mathrm{d}x, x] = \frac{(\Delta x)^2}{\Delta t}\mathrm{d}t \tag{3.9}$$

$$[\mathrm{d}x, t] = [\mathrm{d}t, x] = \Delta t \mathrm{d}x \tag{3.10}$$

$$[\mathrm{d}t, t] = \Delta t \mathrm{d}t. \tag{3.11}$$

4. Discrete Exterior Calculus

In [?], it was demonstrated that if all the commutative relations vanish, we obtain the exterior calculus of standard differential geometry on smooth manifolds. Now, consider the commutative relations corresponding to the binary tree above. If we set $\Delta x = c\Delta t$, we obtain

$$[\mathrm{d}x, x] = c\Delta t \mathrm{d}t \tag{4.1}$$

$$[\mathrm{d}x, t] = [\mathrm{d}t, x] = \Delta t \mathrm{d}x \tag{4.2}$$

$$[\mathrm{d}t, t] = \Delta t \mathrm{d}t. \tag{4.3}$$

In the continuum limit $\Delta t \to 0$, all the relations vanish. Hence, this binary tree has the continuum exterior calculus as its continuum limit. This would be of interest if you were modelling hyperbolic partial differential equations.

5. Discrete Stochastic Calculus

For the purposes of this paper, the more interesting case is obtained when we set $\Delta t = (\Delta x)^2$. In this case, the commutative relations reduce to

$$[\mathrm{d}x, x] = \mathrm{d}t \tag{5.1}$$

$$[\mathrm{d}x, t] = [\mathrm{d}t, x] = \Delta t \mathrm{d}x \tag{5.2}$$

$$[\mathrm{d}t, t] = \Delta t \mathrm{d}t, \tag{5.3}$$

which have the continuum limit

$$[\mathrm{d}x, x] = \mathrm{d}t \tag{5.4}$$

$$[\mathrm{d}x, t] = [\mathrm{d}t, x] = [\mathrm{d}t, t] = 0. \tag{5.5}$$

In [**?**], these commutative relations were shown to lead to stochastic calculus. Hence, the binary tree obtained from setting $\Delta t = (\Delta x)^2$ has the continuum stochastic calculus as a limit. In this respect, we will refer to the discrete calculus on this special binary tree as the "discrete stochastic calculus." In the following, we will examine some consequences of the discrete version of stochastic calculus.

5.1 Discrete Ito Formula

In Equations (3.7) and (3.8), it was found that

$$\mathrm{d}t(i,j) = \Delta t \left[\mathbf{e}^{(i,j)(i+1,j+1)} + \mathbf{e}^{(i,j)(i-1,j+1)}\right] \tag{5.6}$$

and

$$\mathrm{d}x(i,j) = \Delta x \left[\mathbf{e}^{(i,j)(i+1,j+1)} - \mathbf{e}^{(i,j)(i-1,j+1)}\right]. \tag{5.7}$$

These expressions may be inverted giving

$$\mathbf{e}^{(i,j)(i+1,j+1)} = \frac{1}{2\Delta t}\mathrm{d}t(i,j) + \frac{1}{2\Delta x}\mathrm{d}x(i,j) \tag{5.8}$$

and

$$\mathbf{e}^{(i,j)(i-1,j+1)} = \frac{1}{2\Delta t}\mathrm{d}t(i,j) - \frac{1}{2\Delta x}\mathrm{d}x(i,j). \tag{5.9}$$

The discrete Ito formula arises from a simple application of the discrete exterior derivative to an arbitrary discrete 0-form

$$\begin{aligned}\mathrm{d}\phi = &\sum_{i,j} \left[\phi(i+1,j+1) - \phi(i,j)\right] \mathbf{e}^{(i,j)(i+1,j+1)} \\ &+ \sum_{i,j} \left[\phi(i-1,j+1) - \phi(i,j)\right] \mathbf{e}^{(i,j)(i-1,j+1)}.\end{aligned} \tag{5.10}$$

This is quite simple, but it doesn't look very much like the usual Ito formula. To correct for this, plug in Equations (5.8) and (5.9) into the above. This results in

$$\begin{aligned}
\mathrm{d}\phi &= \sum_{i,j} [\phi(i+1,j+1) - \phi(i,j)]\, \mathbf{e}^{(i,j)(i+1,j+1)} \\
&\quad + \sum_{i,j} [\phi(i-1,j+1) - \phi(i,j)]\, \mathbf{e}^{(i,j)(i-1,j+1)} \\
&= \sum_{i,j} [\phi(i+1,j+1) - \phi(i,j)] \left[\frac{1}{2\Delta t}\mathrm{d}t(i,j) + \frac{1}{2\Delta x}\mathrm{d}x(i,j)\right] \\
&\quad + \sum_{i,j} [\phi(i-1,j+1) - \phi(i,j)] \left[\frac{1}{2\Delta t}\mathrm{d}t(i,j) - \frac{1}{2\Delta x}\mathrm{d}x(i,j)\right] \\
&= \sum_{i,j} \left[\frac{\phi(i+1,j+1) - 2\phi(i,j) + \phi(i-1,j+1)}{2\Delta t}\right] \mathrm{d}t(i,j) \\
&\quad + \sum_{i,j} \left[\frac{\phi(i+1,j+1) - \phi(i-1,j+1)}{2\Delta x}\right] \mathrm{d}x(i,j). \qquad (5.11)
\end{aligned}$$

Although the value $\phi(i, j+1)$ does not exist on our binary tee, let us introduce it along with the following operators:

$$\partial_t \phi = \sum_{i,j} \left[\frac{\phi(i,j+1) - \phi(i,j)}{\Delta t}\right] \mathbf{e}^{(i,j)}, \qquad (5.12)$$

$$\partial_x \phi = \sum_{i,j} \left[\frac{\phi(i+1,j+1) - \phi(i-1,j+1)}{2\Delta x}\right] \mathbf{e}^{(i,j)}, \qquad (5.13)$$

and

$$\partial_x^2 \phi = \sum_{i,j} \left[\frac{\phi(i+1,j+1) - 2\phi(i,j+1) + \phi(i-1,j+1)}{(\Delta x)^2}\right] \mathbf{e}^{(i,j)}. \qquad (5.14)$$

Keeping in mind that we have $\Delta t = (\Delta x)^2$, it is simple to verify that

$$\partial_t \phi + \frac{1}{2}\partial_x^2 \phi = \left[\frac{\phi(i+1,j+1) - 2\phi(i,j) + \phi(i-1,j+1)}{2\Delta t}\right] \mathbf{e}^{(i,j)}. \qquad (5.15)$$

It follows that the Ito formula (5.10) may be expressed in a more familiar form via

$$\mathrm{d}\phi = \left(\partial_t \phi + \frac{1}{2}\partial_x^2 \phi\right) \mathrm{d}t + (\partial_x \phi)\, \mathrm{d}x. \qquad (5.16)$$

Why anyone would want to *use* this form rather than the simple expression in (5.10) beats me.

6. Financial Modelling

That was great, but what does it *buy* us?

Most financial models (that I am aware of anyway) are based on some stochastic processes. What discrete stochastic calculus allows us to do is to write down *any* financial model we like using the continuum version, turn the crank, and out pops a robust numerical algorithm that is guaranteed to provide solutions that converge to the continuum solutions. In fact, with some clever programming you could potentially automate this process. For example, you could enter expressions representing the stochastic process, then the code automatically generates the correct algorithm and provides a solution.

Discrete stochastic calculus provides a kind of *meta* algorithm. It is an algorithm for generating algorithms.

6.1 The Discrete Black-Scholes Model

As a simple application, we will turn the crank on the Black-Scholes model and see what pops out. The solution may not be such a big surprise, but it demonstrates that the process works. This provides reason for optimism that it will work for more complicated models and provides exciting possibilities for future work.

Let V denote the value of an option, B denote the value of a risk-free bond, S denote the value of a stock, Π denote the value of the portfolio, α denote the holding in the option, Δ denote the holding in the stock, and β denote the holding in the bond. The outline of our approach is the following:

1. Compute $\mathrm{d}V$ and $\mathrm{d}B$ and express in terms of $\mathrm{d}S$ and $\mathrm{d}t$.
2. Enforce self-financing so that $\mathrm{d}\Pi = \alpha \mathrm{d}V + \Delta \mathrm{d}S + \beta \mathrm{d}B$.
3. Enforce no arbitrage so that $\Pi(i,j) = \exp(jr\Delta t)\Pi(0,0)$.

6.1.1 Change of Variables

The first step in deriving the discrete Black-Scholes equation is to express everything in terms of the differentials $\mathrm{d}S$ and $\mathrm{d}t$ instead of $\mathrm{d}x$ and $\mathrm{d}t$. To do this, first we compute $\mathrm{d}S$ using the discrete Ito formula (5.10), resulting in

$$\begin{aligned} \mathrm{d}S &= \sum_{i,j} \mathrm{d}S(ij) \\ &= \sum_{i,j} \left[\Delta S^+(i,j)\mathbf{e}^{(i,j)(i+1,j+1)} + \Delta S^-(i,j)\mathbf{e}^{(i,j)(i-1,j+1)}\right], \end{aligned} \tag{6.1}$$

where

$$\Delta S^+(i,j) = S(i+1,j+1) - S(i,j) \tag{6.2}$$

and

$$\Delta S^-(i,j) = S(i-1,j+1) - S(i,j). \tag{6.3}$$

This, together with (3.7) results in

$$\mathbf{e}^{(i,j)(i+1,j+1)} = \frac{1}{\Delta S(i,j)} \left[\mathrm{d}S(i,j) - \frac{\Delta S^-(i,j)}{\Delta t} \mathrm{d}t(i,j) \right] \tag{6.4}$$

and

$$\mathbf{e}^{(i,j)(i-1,j+1)} = \frac{1}{\Delta S(i,j)} \left[-\mathrm{d}S(i,j) + \frac{\Delta S^+(i,j)}{\Delta t} \mathrm{d}t(i,j) \right], \tag{6.5}$$

where

$$\Delta S(i,j) = S(i+1,j+1) - S(i-1,j+1). \tag{6.6}$$

Equations (6.4) and (6.5) allow us to write any differential in terms of dS and dt. Therefore, we may compute

$$\begin{aligned}
\mathrm{d}V &= \sum_{i,j} \left[\Delta V^+(i,j) \mathbf{e}^{(i,j)(i+1,j+1)} + \Delta V^-(i,j) \mathbf{e}^{(i,j)(i-1,j+1)} \right] \\
&= \sum_{i,j} \frac{\Delta V^+(i,j)}{\Delta S(i,j)} \left[\mathrm{d}S(i,j) - \frac{\Delta S^-(i,j)}{\Delta t} \mathrm{d}t(i,j) \right] \\
&\quad + \sum_{i,j} \frac{\Delta V^-(i,j)}{\Delta S(i,j)} \left[-\mathrm{d}S(i,j) + \frac{\Delta S^+(i,j)}{\Delta t} \mathrm{d}t(i,j) \right] \\
&= \sum_{i,j} \left[\frac{\Delta V(i,j)}{\Delta S(i,j)} \right] \mathrm{d}S(i,j) \\
&\quad + \sum_{i,j} \left[-\frac{\Delta S^-(i,j)}{\Delta S(i,j)} \frac{\Delta V^+(i,j)}{\Delta t} \quad + \frac{\Delta S^+(i,j)}{\Delta S(i,j)} \frac{\Delta V^-(i,j)}{\Delta t} \right] \mathrm{d}t(i,j)
\end{aligned} \tag{6.7}$$

and

$$\mathrm{d}B = \sum_{i,j} \frac{\Delta B(i,j)}{\Delta t} \mathrm{d}t(i,j), \tag{6.8}$$

where

$$\Delta B(i,j) = \left[\exp(r\Delta t) - 1 \right] B(i,j). \tag{6.9}$$

6.1.2 Self Financing

To ensure that the portfolio is self financing, we must have

$$\begin{aligned}
\mathrm{d}\Pi &= \alpha \mathrm{d}V + \Delta \mathrm{d}S + \beta \mathrm{d}B \\
&= \sum_{i,j} \left[\alpha(i,j) \frac{\Delta V(i,j)}{\Delta S(i,j)} + \Delta(i,j) \right] \mathrm{d}S(i,j) \\
&\quad + \sum_{i,j} \alpha(i,j) \left[-\frac{\Delta S^-(i,j)}{\Delta S(i,j)} \frac{\Delta V^+(i,j)}{\Delta t} \quad + \frac{\Delta S^+(i,j)}{\Delta S(i,j)} \frac{\Delta V^-(i,j)}{\Delta t} \right] \mathrm{d}t(i,j) \\
&\quad + \sum_{i,j} \beta(i,j) \frac{\Delta B(i,j)}{\Delta t} \mathrm{d}t(i,j)
\end{aligned} \tag{6.10}$$

6.1.3 No Arbitrage

To ensure that there are no arbitrage opportunities, we must have

$$\Pi(i,j) = \Pi(0,0)\exp(jr\Delta t). \tag{6.11}$$

Therefore,

$$\begin{aligned}
\mathrm{d}\Pi &= \sum_{i,j} \frac{\exp(r\Delta t)-1}{\Delta t}\left[\alpha(i,j)V(i,j) + \Delta(i,j)S(i,j) + \beta(i,j)B(i,j)\right]\mathrm{d}t(i,j) \\
&= \sum_{i,j} \frac{\exp(r\Delta t)-1}{\Delta t}\left[\alpha(i,j)V(i,j) + \Delta(i,j)S(i,j)\right]\mathrm{d}t(i,j) \\
&\quad + \sum_{i,j} \beta(i,j)\frac{\Delta B(i,j)}{\Delta t}\mathrm{d}t(i,j)
\end{aligned} \tag{6.12}$$

6.1.4 Δ-Hedging

On comparing (6.10) and (6.12), the first thing to note is that (6.12) has no $\mathrm{d}S(i,j)$ term. Therefore, the $\mathrm{d}S(i,j)$ term in (6.10) must vanish. This gives the hedging strategy

$$\Delta(i,j) = -\alpha(i,j)\frac{\Delta V(i,j)}{\Delta S(i,j)}. \tag{6.13}$$

6.1.5 The Discrete Black-Scholes Equation

Substituting (6.13) back into (6.12) gives

$$\begin{aligned}
\mathrm{d}\Pi &= \sum_{i,j} \alpha(i,j)\frac{\exp(r\Delta t)-1}{\Delta t}\left[V(i,j) - \frac{\Delta V(i,j)}{\Delta S(i,j)}S(i,j)\right]\mathrm{d}t(i,j) \\
&\quad + \sum_{i,j} \beta(i,j)\frac{\Delta B(i,j)}{\Delta t}\mathrm{d}t(i,j)
\end{aligned} \tag{6.14}$$

Equating (6.14) and (6.10) results in the discrete Black-Scholes equation

$$\begin{aligned}
0 &= -\frac{\Delta S^-(i,j)}{\Delta S(i,j)}\frac{\Delta V^+(i,j)}{\Delta t} + \frac{\Delta S^+(i,j)}{\Delta S(i,j)}\frac{\Delta V^-(i,j)}{\Delta t} - \tilde{r}\left[V(i,j) - \frac{\Delta V(i,j)}{\Delta S(i,j)}S(i,j)\right] \\
&= \frac{1}{\Delta t}\left\{q(i,j)V(i+1,j+1) + \left[1-q(i,j)\right]V(i-1,j+1) - \left[\exp(r\Delta t)\right]V(i,j)\right\}.
\end{aligned} \tag{6.15}$$

where

$$\tilde{r} = \frac{\exp(r\Delta t)-1}{\Delta t}. \tag{6.16}$$

and

$$q(i,j) = \frac{S(i,j)\exp(r\Delta t) - S(i-1,j+1)}{S(i+1,j+1) - S(i-1,j+1)} \tag{6.17}$$

The discrete Black-Scholes equation may be solved for $V(i,j)$ resulting in

$$V(i,j) = \exp(-r\Delta t)\left\{q(i,j)V(i+1,j+1) + \left[1-q(i,j)\right]V(i-1,j+1)\right\}. \tag{6.18}$$

Equation (6.18) is the well-known Cox-Ross-Rubinstein solution for the binomial option pricing formula. Hence, the Cox-Ross-Rubinstein equation is equivalent to the discrete Black-Scholes equation.

7. Conclusion

In the present paper, a review of discrete calculus was given. It was shown that for a particular choice of parameters, the discrete calculus on a binary tree gives rise to discrete stochastic calculus. The resulting discrete calculus may be used to construct quite general financial models. As an illustration, the technique was applied to the well known Black-Scholes model. It turned out that the discrete Black-Scholes equation is equivalent to the Cox-Ross-Rubinstein equation, as it should be.

The ideas contained in discrete stochastic calculus open the door to a host of exciting research possibilities.